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Insurance Europe Paper on the convergence and stability of the Smith-Wilson method

Our Reference:	ECO-SLV-12-313	Date:	05 June 2012
Referring to:			
Related documents:			
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Pages:	9		

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1. Introduction

For the purpose of Solvency II a robust extrapolation method is needed for all currencies. Moreover, the stability of the extrapolation method is important for the risk management of a company´s solvency position.

EIOPA proposes in the pre-consultation paper as of 13th of March 2012[i] to make use of the Smith-Wilson method for extrapolation of the interest rate curve. The stability of the Smith-Wilson method depends on the speed of convergence amongst other things. In this note we show several, to the best of our knowledge, new results regarding the Smith-Wilson method:

- Explicit expression for extrapolated discount factors
- Convergence time to a long term discount factor
- Convergence time to long term forward rates
- Sensitivity analysis of the forward rate with respect to the last forward rate
- Stability condition for the Smith-Wilson method in terms of alpha

2. Convergence of the extrapolated curve

In the EIOPA paper it is shown that for the extrapolated part of the curve the Smith-Wilson method simplifies significantly.

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Let f(t) denote the forward rate at time t. Note that this is the instantaneous forward rate calculated using the Smith-Wilson. The last liquid point in time is $\mathbf{u_J}$. The ultimate forward rate (UFR) is denoted as β and the speed of convergence parameter in the Smith-Wilson method is denoted as $\boldsymbol{\alpha}$.

As shown in the EIOPA paper for $\mathbf{t} \geq \mathbf{u_J}$ then

$$\mathbf{f(t)} = \beta + \frac{\boldsymbol{\alpha} \mathbf{e}^{-\alpha(\mathbf{t}-\mathbf{u_J})}(\mathbf{f}(\mathbf{u_J}) - \beta)}{\boldsymbol{\alpha} - (\mathbf{1} - \mathbf{e}^{-\alpha(\mathbf{t}-\mathbf{u_J})})(\mathbf{f}(\mathbf{u_J}) - \beta)}$$

As a consequence the extrapolated forward rate depends only on the instantaneous forward rate at the last liquid point in time. Note that this forward rate is calculated using the Smith-Wilson method and hence depends on earlier liquid forward rates. Furthermore, the extrapolated forward rate depends on UFR, $\boldsymbol{\alpha}$ and time.

From the above expression it follows that an explicit expression can be calculated for the discount factors. Given certain conditions (see appendix) and for $\mathbf{t} \geq \mathbf{u_J}$ then

$$\mathbf{P(t)} = \mathbf{P}(\mathbf{u_J})\mathbf{e}^{-\beta(\mathbf{t}-\mathbf{u_J})}\left(\mathbf{1} - \frac{\mathbf{1}}{\boldsymbol{\alpha}}(\mathbf{1} - \mathbf{e}^{-\alpha(\mathbf{t}-\mathbf{u_J})})(\mathbf{f}(\mathbf{u_J}) - \beta)\right)$$

Proof is in the appendix (proposition 1). Note that if $\mathbf{f}(\mathbf{u_J}) = \beta$ then

$$\mathbf{P(t)} = \mathbf{P}(\mathbf{u_J})\mathbf{e}^{-\beta(\mathbf{t}-\mathbf{u_J})}$$

This equals to $\mathbf{P}_\beta(\mathbf{t}) = \mathbf{e}^{-\beta\mathbf{t}}$ if $\mathbf{P}(\mathbf{u_J}) = \mathbf{e}^{-\beta\mathbf{u_J}}$. Moreover, in general, if $\mathbf{f}(\mathbf{u_J}) \neq \beta$ then P(t) equals $\mathbf{P}_\beta(\mathbf{t})$ for time T where

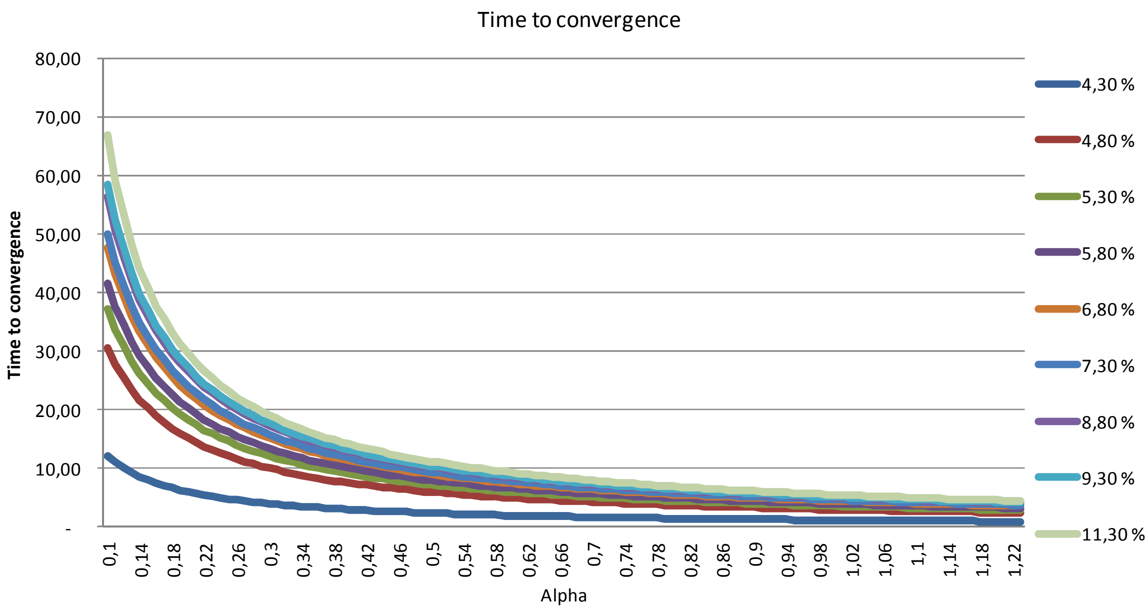
$$T = u_J - \frac{1}{\alpha}\ln\left(1 - \frac{\alpha}{f(u_J) - \beta}\left(1 - \frac{1}{P(u_J)e^{\beta u_J}}\right)\right)$$

Proof is in the appendix (proposition 2). Several discussions in the Solvency II process concern convergence in forward rates, not in the discount factors. Furthermore, a discussion has been that within a given time the extrapolated forward rates should converge to 3 bps within the ultimate forward rate (β). Strictly speaking we are seeking the time where $f(t)$ is very close to β and not necessarily mathematical convergence. However, we use this term as this is used in the Solvency II process. Hence we seek time t such that $|f(t) - \beta| \leq \kappa$ where κ equals for example 3 bps. Here $|x|$ denotes the absolute value of x.

For $\alpha \geq |f(u_J) - \beta| \neq 0$ and $f(u_J) \geq \beta$, then $|f(t) - \beta| \leq \kappa$ if and only if

$$\mathbf{t} \geq \mathbf{u_J} - \frac{\mathbf{1}}{\boldsymbol{\alpha}}\ln\left(\frac{\boldsymbol{\kappa}(\boldsymbol{\alpha} - |\mathbf{f}(\mathbf{u_J}) - \boldsymbol{\beta}|)}{(\boldsymbol{\alpha} - \boldsymbol{\kappa})|\mathbf{f}(\mathbf{u_J}) - \boldsymbol{\beta}|}\right)$$

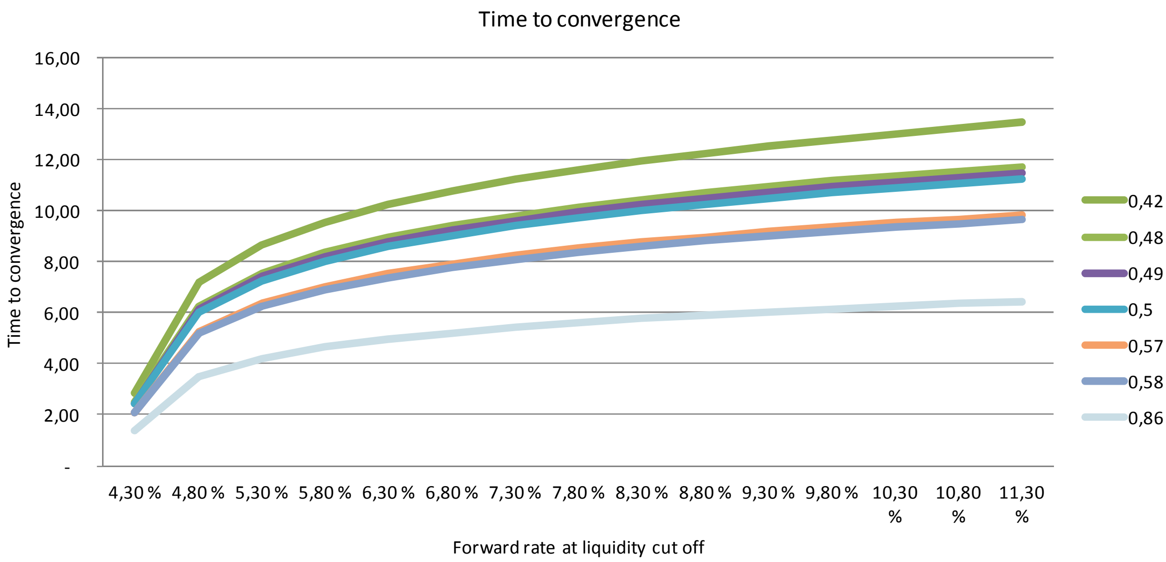
The proof is in the appendix (proposition 3). Below we plot the time to convergence for different alphas and last liquid forward rates. We have here made use of an UFR of 4,2%. This may differ for different currencies.



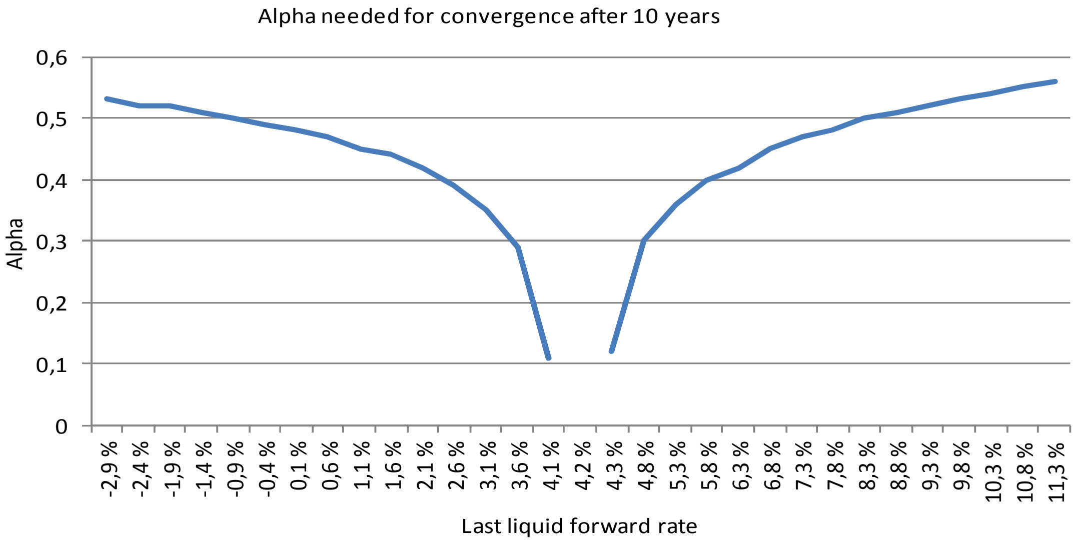
An alternative view is to consider for a given alpha what is the time to convergence for different forward rates.

Note that a similar result can be shown for forward rates less than the UFR. Namely, for $\boldsymbol{\alpha} \geq |\mathbf{f}(\mathbf{u_J}) - \beta| \neq \mathbf{0}$ and $\mathbf{f}(\mathbf{u_J}) \leq \beta$, then $|\mathbf{f}(\mathbf{t}) - \beta| \leq \kappa$ if and only if

$$\mathbf{t} \geq \mathbf{u_J} - \frac{\mathbf{1}}{\boldsymbol{\alpha}} \ln\left(\frac{\kappa(\boldsymbol{\alpha} + |\mathbf{f}(\mathbf{u_J}) - \beta|)}{(\boldsymbol{\alpha} + \kappa)|\mathbf{f}(\mathbf{u_J}) - \beta|}\right)$$



The same can be graphed for lower forward rates than the UFR. Moreover one can plot the alpha needed for a specific convergence. In order to have a convergence after ten years, alpha needs to be in the range 0,15 to 0,6 depending on the last forward rate.



3. Stability of the extrapolated curve

Several parties have pointed out to that the extrapolated curve can be very sensitive to the last liquid forward rate. An interesting question is then how much the extrapolated forward rate changes if the last liquid forward rate changes with ε.

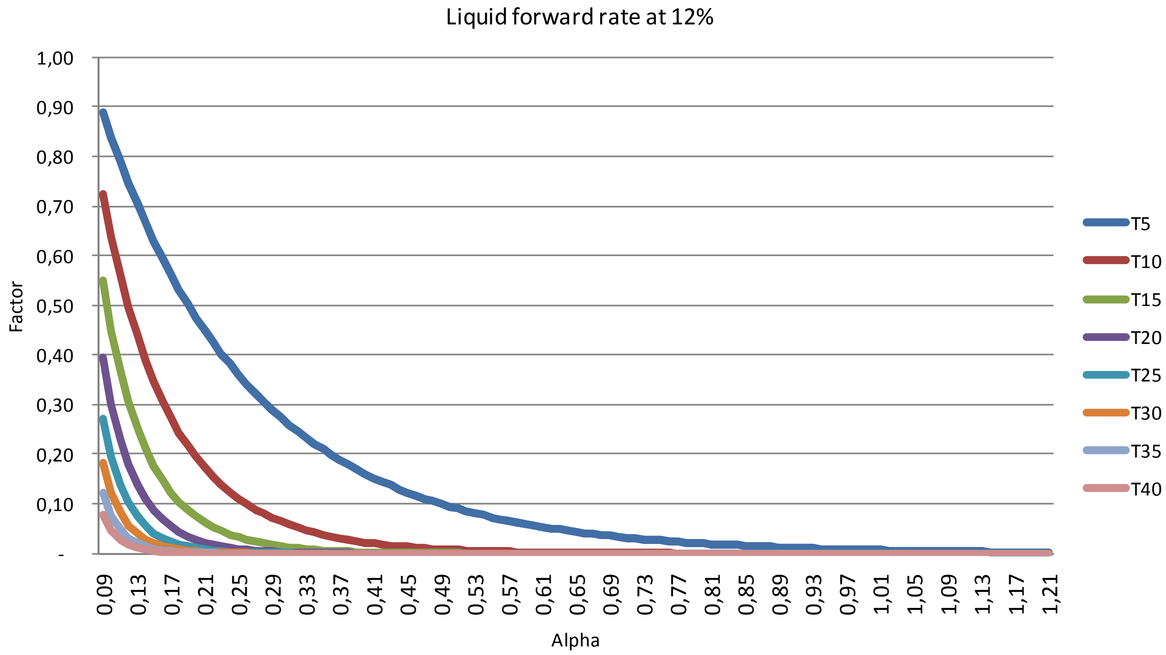
Consider the extrapolated forward rate as a function of the last liquid forward rate x

$$\mathbf{f(t,x)} = \beta + \frac{\boldsymbol{\alpha} \mathbf{e}^{-\alpha(t-u_J)}(\mathbf{x} - \beta)}{\boldsymbol{\alpha} - (\mathbf{1} - \mathbf{e}^{-\alpha(t-u_J)})(\mathbf{x} - \beta)}$$

Then one can show that

$$\mathbf{f(t,x+\varepsilon)} - \mathbf{f(t,x)} = \frac{\boldsymbol{\alpha}^2 \mathbf{e}^{-\alpha(t-u_J)}\boldsymbol{\varepsilon}}{\boldsymbol{\alpha} - (\mathbf{1} - \mathbf{e}^{-\alpha(t-u_J)})(\mathbf{x} - \beta)} \cdot \frac{\mathbf{1}}{\boldsymbol{\alpha} - (\mathbf{1} - \mathbf{e}^{-\alpha(t-u_J)})(\mathbf{x} + \boldsymbol{\varepsilon} - \beta)}$$

Proof is in the appendix (proposition 4). We can then graph this function for different times, forward rates and alpha values. For small alphas the expression is highly unstable and they are omitted in the graphs below. The parameter ε is set to 0,01%. The graph below illustrates that alpha should at least be set to be 0,2 for high forward rates.



Furthermore, the first order expansion is given by

$$f(t,x+\varepsilon)-f(t,x)=\frac{\alpha^2 e^{-\alpha(t-u_J)}}{\left(\alpha-\left(1-e^{-\alpha(t-u_J)}\right)(x-\ \beta)\right)^2}\cdot\varepsilon+\mathcal{O}(\varepsilon^2)$$

where $\mathcal{O}(\varepsilon^2)$ means that the other terms are polynomials of ε^2 or higher order terms. The first derivative of the forward rate with respect to last liquid point is hence given by

$$\frac{\partial f(t,x)}{\partial x}=\lim_{\varepsilon\to 0}\frac{f(t,x+\varepsilon)-f(t,x)}{\varepsilon}=\frac{\alpha^2 e^{-\alpha(t-u_J)}}{\left(\alpha-\left(1-e^{-\alpha(t-u_J)}\right)(x-\ \beta)\right)^2}$$

Let us then define stability of the Smith Wilson be that the first derivative should be bounded by 1. In other words as an first order approximation the extrapolated forward rate should not change more than the last liquid forward rate.

For a given alpha, the stability must hold for all times t. The stability will depend on the difference between the last forward rate and the UFR. We define stability by

$$\left|\frac{\partial f(t,x)}{\partial x}\right|\leq 1$$

for all time t, if $x \geq \beta$ then we can show that the Smith-Wilson method is stable if and only if alpha obeys

$$\left(1+e^{-\frac{\alpha}{2}(t-u_J)}\right)|x-\ \beta|\leq\alpha$$

Since this should be valid for all time t, a criteria for alpha is given by:

$$|x - \beta| \cdot 2 \leq \alpha$$

The proof is in the appendix (proposition 5). As an example if forward rates are 10% higher than the ultimate forward rate (4,2%), then alpha needs to be more than 0,2 from a stability perspective.

However, it should be noted that this stability condition is not necessarily sufficient because it may be the last instantaneous forward rate $\mathbf{f}(\mathbf{u_J})$ that is unstable due to the Smith Wilson method. It would not then help that all later forward rates are lower than an unnaturally high $\mathbf{f}(\mathbf{u_J})$. This is the case in the examples shown in the CEA paper on "Avoiding artificial volatility in extrapolation"[ii]. Further work is therefore needed to investigate further on the stability of $\mathbf{f}(\mathbf{u_J})$. Note that a rapid convergence would reduce this problem.

4. Conclusion

The extrapolated curve should be predictable over time**.** Therefore, the extrapolation method should be such that the extrapolated part of the curve is stable over time. In particular, a small change in the forward rates of the liquid part of the curve should not produce large changes in the extrapolated part of the curve. Furthermore an argument in the Solvency II process has been that the extrapolation should converge rapidly to the ultimate forward rate.

In this note we have shown several results related to the Smith Wilson method which are of interest:

- Explicit expression for extrapolated discount factors
- Convergence time to a long term discount factor
- Convergence time to long term forward rates
- Sensitivity analysis of the forward rate with respect to the last forward rate
- Stability condition for the Smith and Wilson method in terms of alpha

From a stability perspective alpha should be larger than 0,2 for high forward rates. However, if a convergence in forward rates is needed after ten years. Alpha should be higher (0,5 to 0,6) for forward rates above 8%. Based on an empirical probability on liquid forward rates one could propose that alpha should be for example 0,5 so that the Smith-Wilson method converges after 10 years with x% probability.

Further work needs to be done on the stability on the instantaneous forward rate at the last liquid point. Instability of this forward rate due to the Smith Wilson method could produce artificial volatility of the extrapolated curve. A more rapid convergence would be a partial remedy to this problem.

5. Appendix: propositions and proofs

Proposition 1: Explicit expression for extrapolated discount factor

If $\alpha \geq f(u_J) - \beta > 0$ then for $t \geq u_J$ we have

$$P(t) = P(u_J)e^{-\beta(t-u_J)}\left(1 - \tfrac{1}{\alpha}\left(1 - e^{-\alpha(t-u_J)}\right)\left(f(u_J) - \beta\right)\right)$$

Proof of proposition 1:

$$P(t) = e^{-\int_0^t f(s)ds} = P(u_J)e^{-\int_{u_J}^t f(s)ds}$$

$$P(t) = P(u_J)e^{-\beta(t-u_J)}e^{-\int_{u_J}^t \frac{\alpha e^{-\alpha(s-u_J)}(f(u_J)-\beta)}{\alpha-\left(1-e^{-\alpha(s-u_J)}\right)(f(u_J)-\beta)}ds} = P(u_J)e^{-\beta(t-u_J)}e^{\left[\ln\left(\alpha-\left(1-e^{-\alpha(s-u_J)}\right)(f(u_J)-\beta)\right)\right]_{u_J}^t}$$

$$P(t) = P(u_J)e^{-\beta(t-u_J)}e^{\ln\left(\alpha-\left(1-e^{-\alpha(t-u_J)}\right)(f(u_J)-\beta)\right)-\ln(\alpha)} = P(u_J)e^{-\beta(t-u_J)}\left(1 - \tfrac{1}{\alpha}\left(1 - e^{-\alpha(t-u_J)}\right)\left(f(u_J) - \beta\right)\right)$$

Can make use of ln since $\alpha - \left(1 - e^{-\alpha(s-u_J)}\right)\left(f(u_J) - \beta\right) \geq e^{-\alpha(s-u_J)}\left(f(u_J) - \beta\right) > 0$

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Proposition 2: Extrapolated discount factor

If $\alpha \geq f(u_J) - \beta > 0$ and given $P_\beta(t) = e^{-\beta t}$ and if $f(u_J) \neq \beta$ then P(t) equals $P_\beta(t)$ at time T if

$$T = u_J - \tfrac{1}{\alpha}\ln\left(1 - \frac{\alpha}{f(u_J) - \beta}\left(1 - \frac{1}{P(u_J)e^{\beta u_J}}\right)\right)$$

Proof of proposition 2:

P(t) equals $P_\beta(t)$ at time T if P(T)= $P_\beta(T)$

$$P(T) = P(u_J)e^{-\beta(T-u_J)}\left(1 - \tfrac{1}{\alpha}\left(1 - e^{-\alpha(T-u_J)}\right)\left(f(u_J) - \beta\right)\right) = e^{-\beta T}$$

$$\Leftrightarrow \tfrac{1}{\alpha}\left(1 - e^{-\alpha(T-u_J)}\right)\left(f(u_J) - \beta\right) = 1 - \frac{1}{P(u_J)e^{\beta u_J}}$$

$$\Leftrightarrow e^{-\alpha(T-u_J)} = 1 - \frac{\alpha}{f(u_J) - \beta}\left(1 - \frac{1}{P(u_J)e^{\beta u_J}}\right)$$

The last equivalence is true since $f(u_J) \neq \beta$. The result follows by taking ln on both sides and rearranging terms. ■

Proposition 3: Convergence of extrapolated forward rates

For $\alpha \geq |f(u_J) - \beta|$, $f(u_J) \geq \beta$ and $0 < \kappa < \alpha$, then $|f(t) - \beta| \leq \kappa$ if and only if

$$t \geq u_J - \frac{1}{\alpha}\ln\left(\frac{\kappa\left(\alpha - |f(u_J) - \beta|\right)}{(\alpha - \kappa)|f(u_J) - \beta|}\right)$$

Proof of proposition 3:

$$|f(t) - \beta| = \left|\frac{\alpha e^{-\alpha(t-u_J)}\left(f(u_J) - \beta\right)}{\alpha - (1 - e^{-\alpha(t-u_J)})\left(f(u_J) - \beta\right)}\right| \leq \kappa$$

$$\Leftrightarrow \left|\alpha e^{-\alpha(t-u_J)}\left(f(u_J) - \beta\right)\right| \leq \kappa \cdot \left|\alpha - (1 - e^{-\alpha(t-u_J)})\left(f(u_J) - \beta\right)\right|$$

$$\Leftrightarrow \alpha e^{-\alpha(t-u_J)}\left|f(u_J) - \beta\right| \leq \kappa \cdot \left(\alpha - (1 - e^{-\alpha(t-u_J)})\left|f(u_J) - \beta\right|\right)$$

$$\Leftrightarrow e^{-\alpha(t-u_J)} \cdot \left(\alpha \cdot \left|f(u_J) - \beta\right| - \kappa \cdot \left|f(u_J) - \beta\right|\right) \leq \kappa \cdot \left(\alpha - \left|f(u_J) - \beta\right|\right)$$

$$\Leftrightarrow e^{-\alpha(t-u_J)} \leq \frac{\kappa\left(\alpha - \left|f(u_J) - \beta\right|\right)}{(\alpha - \kappa)\left|f(u_J) - \beta\right|} \Leftrightarrow t \geq u_J - \frac{1}{\alpha}\ln\left(\frac{\kappa\left(\alpha - \left|f(u_J) - \beta\right|\right)}{(\alpha - \kappa)\left|f(u_J) - \beta\right|}\right)$$

Note that $\left|\alpha - (1 - e^{-\alpha(t-u_J)})\left(f(u_J) - \beta\right)\right| = \left(\alpha - (1 - e^{-\alpha(t-u_J)})\left|f(u_J) - \beta\right|\right)$ since $\alpha \geq \left|f(u_J) - \beta\right|$ and $f(u_J) \geq \beta$

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Proposition 4: Sensitivity of extrapolated forward rates to change in the last liquid forward rate

Given ε, one can show that the sensitivity of the forward rates to change in the last liquid forward rate is given by

$$f(t, x + \varepsilon) - f(t, x) = \frac{\alpha^2 e^{-\alpha(t-u_J)}\varepsilon}{\alpha - (1 - e^{-\alpha(t-u_J)})(x - \beta)} \cdot \frac{1}{\alpha - (1 - e^{-\alpha(t-u_J)})(x + \varepsilon - \beta)}$$

Proof of proposition 4:

We have

$$f(t, x) = \beta + \frac{\alpha e^{-\alpha(t-u_J)}(x - \beta)}{\alpha - (1 - e^{-\alpha(t-u_J)})(x - \beta)}$$

Hence

$$f(t, x + \varepsilon) - f(t, x) = \frac{\alpha e^{-\alpha(t-u_J)}(x + \varepsilon - \beta)}{\alpha - (1 - e^{-\alpha(t-u_J)})(x + \varepsilon - \beta)} - \frac{\alpha e^{-\alpha(t-u_J)}(x - \beta)}{\alpha - (1 - e^{-\alpha(t-u_J)})(x - \beta)}$$

$$= \frac{\alpha e^{-\alpha(t-u_J)}(x + \varepsilon - \beta)\left(\alpha - (1 - e^{-\alpha(t-u_J)})(x - \beta)\right) - \alpha e^{-\alpha(t-u_J)}(x - \beta) \cdot \left(\alpha - (1 - e^{-\alpha(t-u_J)})(x + \varepsilon - \beta)\right)}{\left(\alpha - (1 - e^{-\alpha(t-u_J)})(x + \varepsilon - \beta)\right) \cdot \left(\alpha - (1 - e^{-\alpha(t-u_J)})(x - \beta)\right)}$$

$$= \frac{\alpha e^{-\alpha(t-u_J)} \cdot (\varepsilon)\left(\alpha - (1 - e^{-\alpha(t-u_J)})(x - \beta)\right) + \alpha e^{-\alpha(t-u_J)}(x - \beta) \cdot \left((1 - e^{-\alpha(t-u_J)})(\varepsilon)\right)}{\left(\alpha - (1 - e^{-\alpha(t-u_J)})(x + \varepsilon - \beta)\right) \cdot \left(\alpha - (1 - e^{-\alpha(t-u_J)})(x - \beta)\right)}$$

$$= \frac{\alpha^2 e^{-\alpha(t-u_J)}\varepsilon}{\alpha - (1 - e^{-\alpha(t-u_J)})(x - \beta)} \cdot \frac{1}{\alpha - (1 - e^{-\alpha(t-u_J)})(x + \varepsilon - \beta)}$$

■

Proposition 5: Stability of the Smith-Wilson method

Let $x \geq \beta$, then the Smith-Wilson method is stable for extrapolated forward rates at time t if and only if alpha obeys

$$\left(1 + e^{-\frac{\alpha}{2}(t-u_J)}\right)|x - \beta| \leq \alpha$$

Moreover for all time the Smith Wilson method is stable if

$$2|x - \beta| \leq \alpha$$

Proof of proposition 5:

$$\left|\frac{\partial f(t,x)}{\partial x}\right| \leq 1 \Leftrightarrow \frac{\alpha^2 e^{-\alpha(t-u_J)}}{\left(\alpha - \left(1 - e^{-\alpha(t-u_J)}\right)(x - \beta)\right)^2} \leq 1$$

$$\Leftrightarrow \alpha e^{-\frac{\alpha}{2}(t-u_J)} \leq \alpha - \left(1 - e^{-\alpha(t-u_J)}\right)|x - \beta|$$

$$\Leftrightarrow \left(1 - e^{-\alpha(t-u_J)}\right)|x - \beta| \leq \alpha\left(1 - e^{-\frac{\alpha}{2}(t-u_J)}\right)$$

$$\Leftrightarrow \left(1 - e^{-\frac{\alpha}{2}(t-u_J)}\right)\left(1 + e^{-\frac{\alpha}{2}(t-u_J)}\right)|x - \beta| \leq \alpha\left(1 - e^{-\frac{\alpha}{2}(t-u_J)}\right)$$

$$\Leftrightarrow \left(1 + e^{-\frac{\alpha}{2}(t-u_J)}\right)|x - \beta| \leq \alpha$$

A condition is that $\alpha - \left(1 - e^{-\alpha(t-u_J)}\right)(x - \beta) \geq 0$ but

$$\alpha - \left(1 - e^{-\alpha(t-u_J)}\right)(x - \beta) \geq \left(1 + e^{-\frac{\alpha}{2}(t-u_J)}\right)(x - \beta) - \left(1 - e^{-\alpha(t-u_J)}\right)(x - \beta)$$

$$= \left(e^{-\frac{\alpha}{2}(t-u_J)} + e^{-\alpha(t-u_J)}\right)(x - \beta) \geq 0$$

if $x \geq \beta$

The result should hold for all t, and for t=u_J, we have $2|x - \beta| \leq \alpha$

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[i] EIOPA, Determination of the risk free interest rate term structure for Solvency II, 13 March 2012.

[ii] CEA (now Insurance Europe), Avoiding artificial volatility in extrapolation, ECO-11-270, 26 Oct 2011.